MTH 203 - Quiz 3 Solutions

1. Let σ be the n-cycle $(1\,2\,\ldots\,n)$. Then show that σ^i is a n-cycle if and only if $\gcd(i,n) = 1$.

   Solution. By 5.4 (iv) of the Lesson Plan, we know that $o(\sigma) = n$. Thus, it follows that $\langle\sigma\rangle \cong \mathbb{Z}_n$ via the isomorphism $\sigma^i \overset{\varphi}{\mapsto} [i]$, for $0 \leq i \leq n-1$ (Verify this!). Thus, we have:

$$\begin{array}{rcll} \sigma^i \text{ is an } n\text{-cycle} & \Longleftrightarrow & o(\sigma^i) = n & (\text{By 5.4 (iv) and } \sigma^i(j) = i + j \mod n) \\ & \Longleftrightarrow & o(\varphi(\sigma^i)) = n & (\text{By 3.2 (vii)}) \\ & \Longleftrightarrow & o([i]) = n & (\text{By definition of } \varphi) \\ & \Longleftrightarrow & \gcd(i,n) = 1, & (\text{By 1.4 (v)}) \end{array}$$

   and the assertion follows.

2. Show that A_4 has no subgroup of index 2.

   Solution. Since $|A_4| = 12$, any subgroup H of A_4 of index 2 has to be of order 6. Thus, it follows that $H \cong \mathbb{Z}_6$ (or $D_6 = S_3$) (Verify this! See also practice problem no.3 in Homework VII).

   Suppose that $H \cong \mathbb{Z}_6$. Then $H = \langle\sigma\rangle$, where $o(\sigma) = 6$. So, it follows from Lesson Plan 5.4 (iv) that either σ is a 6-cycle or the product of a 3-cycle with a transposition. Since A_4 has no tranpositions or 6-cycles (why?), this is impossible.

   Now, we assume that $H \cong S_3 (= D_6)$. Since A_4 has no tranpositions and only 4 symbols, H would have to contain three distinct elements of order 2 (as in D_6) which have to necessarily arise as products of two disjoint transpositions. Note that the identity permutation together with these 3 elements of order 2 form a subgroup of order 4 inside A_4 (Verify this! See also Problem no.1 in Homework VII.) and hence inside H. This is impossible since $|H| = 6$.